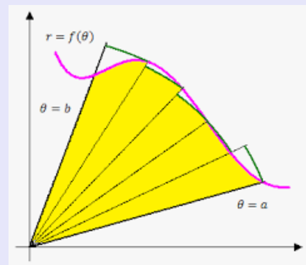
Calculus II

Lecture 3



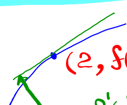
Feb 19-8:47 AM

Class QZ 2

Find the equation of the tangent line to the graph $\boxed{f(x) = \ln(x-1)}$ at $\boxed{x=2}$. Final answer in Slope-Int. form

Hint: $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$

$f(2) = \ln(2-1)$
$= \ln 1$
$= 0$



$(2, f(2)) = (2, 0)$

$m = f'(2) = \frac{1}{2-1} = 1$

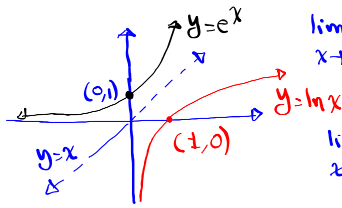
$f'(x) = \frac{1}{x-1} \cdot 1$

$f'(x) = \frac{1}{x-1}$

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x-2) \longrightarrow \boxed{y = x-2}$

$f(x) = \ln x$ is the inverse of $g(x) = e^x$



$\lim_{x \to \infty} e^x = \infty$, $\lim_{x \to -\infty} e^x = 0$

$y = 0$ H.A.

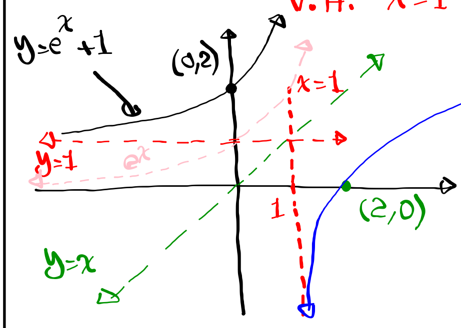
$\lim_{x \to \infty} \ln x = \infty$

$\lim_{x \to 0^+} \ln x = -\infty$

what about $f(x) = \ln(x-1)$

Domain $x-1>0 \to x>1 \to (1,\infty)$

V.A. $x=1$



Take $y=\ln x$, shift right 1 unit

find $f^{-1}(x)$

$f(x) = \ln(x-1)$

$y = \ln(x-1)$

$x = \ln(y-1)$

$x = \ln(y-1)$

$\log_e(y-1) = x$

$e^x = y-1$ $\quad$ $e^x + 1 = y$

$f^{-1}(x) = e^x + 1$

shift up

Some Properties of log:

1) $\log_b x$, $x>0$, $b>0$, $b \neq 1$

2) $b=10 \to$ Common log

$b = e \to$ Natural log

3) $\log_b 1 = 0$, $\log_b b = 1$

4) $\log_b MN = \log_b M + \log_b N$

5) $\log_b \frac{M}{N} = \log_b M - \log_b N$

6) $\log_b M^P = P \log_b M$

7) change-of-base formula $\log_N M = \dfrac{\log_b M}{\log_b N}$

8) $b^{\log_b x} = x$

Simplify / Expand $\log_5 25x^3$

$= \log_5 25 + \log_5 x^3$

$= \log_5 5^2 + \log_5 x^3$

$= 2\boxed{\log_5 5} + 3\log_5 x$

$= \boxed{2 + 3\log_5 x}$

Write as a single log

$4 + \frac{1}{2}\log_2 x - \frac{1}{3}\log_2 y$

$= 4 \cdot 1 + \log_2 x^{1/2} - \log_2 y^{1/3}$

$= 4\log_2 2 + \log_2 \sqrt{x} - \log_2 \sqrt[3]{y}$

$= \log_2 2^4 \rightarrow 16 + \log_2 \sqrt{x} - \log_2 \sqrt[3]{y} = \log_2 \frac{16\sqrt{x}}{\sqrt[3]{y}}$

Use Your Calc, round to 3-decimal places

1) $\log_{10} 2020 \approx 3.305$ verify $10^{3.305} \approx 2020$

2) $\ln 2024 \approx 7.613$ verify $e^{7.613} \approx 2024$

$\log_e 2024 \approx 7.613$ $2.718^{7.613} \approx 2024$

3) $\log_3 81 = ?$ $3^? = 81 \rightarrow 4$

using change-of-base formula

$$\log_3 81 = \frac{\ln 81}{\ln 3} = \frac{\log 81}{\log 3} = 4$$

Find $\frac{dy}{dx}$ if $y = 4^x$ $\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}$

Take ln of both sides

$\ln y = \ln 4^x$

$\ln y = x \ln 4$

Take derivative of both sides

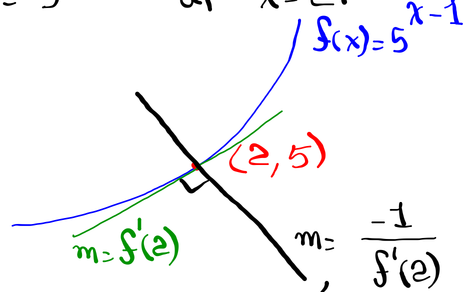
$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[x \ln 4\right]$

$\frac{1}{y}\frac{dy}{dx} = \ln 4 \frac{dx}{dx}$

$\frac{1}{y}\frac{dy}{dx} = \ln 4$ $\frac{dy}{dx} = \ln 4 \cdot y$

$\boxed{\frac{dy}{dx} = \ln 4 \cdot 4^x}$

Find slope of the $\boxed{\text{normal line}}$ to the graph of $f(x) = 5^{x-1}$ at $x = 2$.



$y = 5^{x-1}$

$\ln y = \ln 5^{x-1}$

$\ln y = (x-1) \cdot \ln 5$

$\dfrac{1}{y} \dfrac{dy}{dx} = \ln 5 \cdot 1$

$f'(x) = \ln 5 \cdot 5^{x-1}$

$f'(2) = \ln 5 \cdot 5^{1}$

$= 5 \ln 5$

$$\boxed{m_{\text{Normal}} = \frac{-1}{5 \ln 5}}$$

Find $\dfrac{dy}{dx}$ if $y = (x-2)^3 (x+4)^5$

$y' = 3(x-2)^2 \cdot 1 \cdot (x+4)^5 + (x-2)^3 \cdot 5(x+4)^4 \cdot 1$

$y' = 3(x-2)^2 \cdot (x+4)^5 + 5(x-2)^3 (x+4)^4$

$y' = (x-2)^2 (x+4)^4 \left[3(x+4) + 5(x-2) \right]$

$y' = (x-2)^2 (x+4)^4 (8x + 2)$

$y' = 2(x-2)^2 (x+4)^4 (4x+1)$

$y = (x-2)^3(x+4)^5$

$\ln y = \ln\left[(x-2)^3(x+4)^5\right]$

$\ln y = 3\ln(x-2) + 5\ln(x+4)$

$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x-2} + \frac{5}{x+4}$

$\frac{dy}{dx} = (x-2)^3(x+4)^5\left[\frac{3}{x-2} + \frac{5}{x+4}\right]$

$= 3(x-2)^2(x+4)^5 + 5(x-2)^3(x+4)^4$

$= (x-2)^2(x+4)^4\left[3(x+4) + 5(x-2)\right]$

$\vdots$

See Last Slide

Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{x^2-1}}{\sqrt[3]{x^4+1}}$

$\ln y = \ln\sqrt{x^2-1} - \ln\sqrt[3]{x^4+1}$

$\ln y = \frac{1}{2}\ln(x^2-1) - \frac{1}{3}\ln(x^4+1)$

$\frac{1}{y}\, y' = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2-1} - \frac{1}{3}\cdot\frac{4x^3}{x^4+1}$

$$\boxed{y' = \frac{\sqrt{x^2-1}}{\sqrt[3]{x^4+1}}\left[\frac{x}{x^2-1} - \frac{4x^3}{3(x^4+1)}\right]}$$

Evaluate $\int \tan x \, dx$ for $\boxed{0 < x < \frac{\pi}{2}}$

Recall $\tan x = \dfrac{\sin x}{\cos x}$

$\cos x > 0$
$\sin x > 0$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$$= \int \frac{-1}{u} du = -1 \int \frac{1}{u} du$$

$$= -\ln u + C = \ln u^{-1} + C$$

$$= \ln \frac{1}{u} + C$$

$$= \ln \frac{1}{\cos x} + C$$

$$= \ln(\sec x) + C$$

derive a formula for $\frac{d}{dx}[\log_a x]$

we know $\frac{d}{dx}[\ln x] = \frac{1}{x}$, we also know

$\log_N M = \dfrac{\ln M}{\ln N}$ change-of-base formula

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}[\ln x]$$

$$= \frac{1}{\ln a} \cdot \frac{1}{x}$$

Derive a formula for $\frac{d}{dx}[a^x]$

$y = a^x$

$\ln y = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = \ln a \cdot y = \ln a \cdot a^x$

$$\boxed{\frac{d}{dx}[a^x] = \ln a \cdot a^x}$$

find $\frac{dy}{dx}$ if $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$\frac{dy}{dx} = y\left[\ln x + 1\right]$

$$\boxed{\frac{dy}{dx} = x^x\left[\ln x + 1\right]}$$

Possible attempt

$y' = x \cdot x^{x-1}$

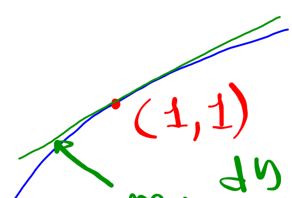
Power Rule

Error, Power Rule exponent must be a real #

find eqn of the tangent line to the graph of $y = x^{\sqrt{x}}$ at $x = 1$.

$y = x^{\sqrt{x}}$

$\ln y = \sqrt{x} \ln x$



$m = \left.\frac{dy}{dx}\right|_{(1,1)}$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}$

$\frac{1}{1}\, m = \frac{1}{2}\cancel{\ln 1}^{\,0} + \frac{1}{1}$

$m = 1$

$y - 1 = 1(x - 1)$

$\boxed{y = x}$

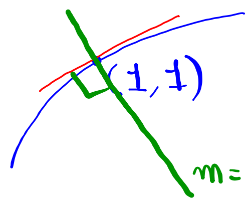
Find eqn of the normal line to the graph of $y = \sqrt{x^x}$ at $x = 1$.

$$y = \sqrt{x^x}$$

$$y^2 = x^x$$

$$\ln y^2 = \ln x^x$$

$$2\ln y = x \ln x$$



$$m = \frac{-1}{\left.\frac{dy}{dx}\right|_{(1,1)}} = \frac{-1}{1/2} = -2$$

$$2 \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$2 \cdot \frac{1}{1} \cdot \left.\frac{dy}{dx}\right|_{(1,1)} = \cancel{1 \cdot \ln 1}^{\,0} + 1 \cdot \frac{1}{1} \qquad \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{2}$$

$$y - 1 = -2(x - 1) \longrightarrow \boxed{y = -2x + 3}$$

Find y' for $y = \dfrac{e^{-x}\cos^2 x}{x^2 + x + 1}$

$$\ln y = \ln \frac{e^{-x}\cos^2 x}{x^2 + x + 1}$$

$$\ln y = \ln e^{-x}\cos^2 x - \ln(x^2 + x + 1)$$

$$\ln y = \ln e^{-x} + \ln \cos^2 x - \ln(x^2 + x + 1)$$

$$\ln y = -x \cancel{\ln e}^{\,1} + 2\ln \cos x - \ln(x^2 + x + 1)$$

$$\frac{1}{y} \cdot y' = -1 + 2 \cdot \frac{-\sin x}{\cos x} - \frac{2x + 1}{x^2 + x + 1}$$

$$y' = y\left[-1 - 2\tan x - \frac{2x + 1}{x^2 + x + 1}\right]$$

$$y' = \frac{-e^{-x}\cos^2 x}{x^2 + x + 1}\left[1 + 2\tan x + \frac{2x + 1}{x^2 + x + 1}\right]$$

Evaluate

$$\int \frac{\sin 2x}{1+\cos^2 x}\,dx$$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\sin 2x = 2\sin x \cos x$

$u = 1 + \cos^2 x$

$du = 2\cos^{1} x \cdot -\sin x\,dx$

$du = -2\cos x \sin x\,dx$

$du = -\sin 2x\,dx$

$-du = \sin 2x\,dx$

$$\int \frac{-1}{u}\,du$$

$$= -\int \frac{1}{u}\,du = -\ln u + C$$

$$= -\ln(1+\cos^2 x) + C$$

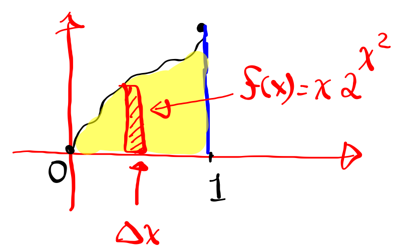
Abs. value not needed, why? $1+\cos^2 x > 0$

Find the area below $f(x) = x \cdot 2^{x^2}$, above x-axis, from $x=0$ to $x=1$.

$f(0) = 0 \cdot 2^{0^2} = 0$

$f(1) = 1 \cdot 2^{1^2} = 2$

$f(x) \geq 0$ on $[0,1]$



$$A = \int_0^1 x\, 2^{x^2}\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \int_0^1 2^u \frac{du}{2}$$

$$= \frac{1}{2}\int_0^1 2^u\,du = \boxed{}$$

Topic from Cal. I

If $f(x) = \int_{u(x)}^{v(x)} g(t)\,dt$, then

$$f'(x) = g(v(x)) \cdot v'(x) - g(u(x)) \cdot u'(x)$$

$$f(x) = \int_1^{x^2} \sqrt{\sin t + \cos t}\, dt$$

$$f'(x) = \sqrt{\sin x^2 + \cos x^2} \cdot 2x - \sqrt{\sin 1 + \cos 1} \cdot 0$$

$$\boxed{f'(x) = 2x\sqrt{\sin x^2 + \cos x^2}}$$

$\sin^2 x + \cos^2 x = 1$

$$f(x) = \int_{\underset{u(x)}{x^3}}^{\overset{v(x)}{x^5}} \underbrace{\sin\sqrt[15]{t}}_{g(t)}\, dt \quad , \text{ Find } f'(x)$$

$$f'(x) = \sin\sqrt[15]{x^5} \cdot 5x^4 - \sin\sqrt[15]{x^3} \cdot 3x^2$$

$$\boxed{f'(x) = 5x^4 \sin\sqrt[3]{x} - 3x^2 \sin\sqrt[5]{x}}$$

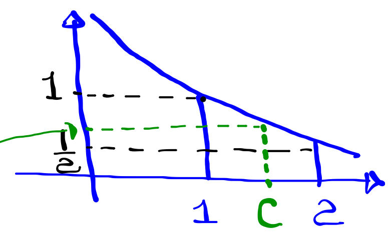
f_{ave} of the cont. function $f(x)$ on $[a,b]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Find f_{ave} for $f(x) = \frac{1}{x}$ on $[1,2]$.

$$f_{ave} = \frac{1}{2-1}\int_1^2 \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^2 = \boxed{\ln 2}$$



Class QZ 3

Evaluate $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

$u = e^x + 1$

$du = e^x\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|e^x+1| + C$$

Since $e^x + 1 > 0$, No Abs. Value needed.

$$= \boxed{\ln(e^x+1) + C}$$